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ABSTRACT

This research involved the observation and audio-recording of 30 ninth-grade mathematics students recommended as being outstanding by their mathematics teachers. Each student was presented with five different problem situations for which he or she was to make mathematical conjectures. During the interview, the students thought aloud and all of their manipulative moves were recorded. The primary focus of this paper is an analysis of creativity factors exhibited by the five highest-scoring students on a particular geometric problem. The comparisons and analyses showed that these five students had much in common even though each student was highly individualistic. (MP)

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PROCESSES AND HEURISTICS INVOLVED IN MATHEMATICAL DIVERGENT PROBLEM SOLVING: A FURTHER ANALYSIS

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This paper is a report of some further analysis* of a portion of data given in Brandau and Dossey (1979). A brief summary of the total study is given below.

The research involved the observation and audio-recording of 30 ninth-grade mathematics students recommended as being outstanding by their mathematics teachers. Each of the students was interviewed individually for about one hour. During this time, the student was presented with five different problem situations for which he/she was to make mathematical conjectures. (See Appendix for a copy of the directions.) During the interview, the students thought aloud and I, as interviewer, recorded any of their manipulative moves.

The audiotapes were transcribed and nonverbal behaviors were interwoven with verbal statements. The transcripts were used in several ways: (a) the protocols were coded using a 40 symbol system which was a modification of Kilpatrick's (1968) and Lucas' system (1974) and (b) the conjectures for each problem were scored for the creativity factors of fluency, flexibility, originality, and organization.

Further analysis was done for one of the problems. The problem chosen is a geometric one. Each student was given one each of the five regular polyhedra. These are: (a) the tetrahedron (called here number 1), (b) the octahedron (called number 2), (c) a cube (called number 3), (d) an icosahedron (called number 4), and (e) a dodecahedron (called number 5). The polyhedra were related in that the edges of each of numbers 1, 2, 3, and 4 were equivalent. (Each was made of styrofoam.)

Further analysis (relating to the Polyhedra problem) was performed for the five "highest scoring students" on the creativity factors. These five

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* The Brandau and Dossey (1979) paper is based on Brandau's masters thesis from December, 1977, supervised by Dossey. This further analysis was done by Brandau in June, 1978.

students (numbered and referred to here as 8, 11, 20, 28, and 29) were chosen because in the thesis analysis, they were most often in each high creativity group. "High" was defined as being one standard deviation above the mean for each score for the creativity factors. Because some of the statistical analysis showed differences between these students and the other students, I felt that further analysis of some of their work would yield some interesting results. (A copy of each of these five student's transcript for the Polyhedra problem is provided here in the Appendix.)

PROCESSES

First some analysis was done relating to the process codes. Out of the 40 codes used, 11 of them were chosen because they involve ideas related to Polya-type heuristics (1957).

These process codes are given below along with the number of times each was used across all 30 students in the Polyhedra problem. Each percent is a comparison of this number to 1914 (the total number of processes used by all 30 students in the Polyhedra problem).

1. An - The student uses an analogy. (4, .2%)

2. Ag - The student assigns a name or meaning to something, similar to Polya's idea of defining. (27, 1.4%)

3. P - The student describes a plan or procedure or tells what he/she is doing at that moment. (25, 1.3%)

4. Da - The student states that some data that would be needed are not provided. (10, .5%)

5. R - The student searches for a meaning of, or for some relevance of the data. (4, .2%)

6. D - The student makes a generalization after reasoning deductively. He/she has posed a theory via the formulation of a problem and then has tried to see if that theory is true. (1, 0%)

7. G - The student makes a generalization, other than those given by induction or deduction. (33, 1.7%)

8. I - The student makes a generalization after reasoning by induction through a trial-and-error type of search. (4, .2%)

9. A - The student adds data to the problem or extends a table. (?,0%)

10. M - The student states he/she is going to make a table or draw a diagram or write something on paper. (?,0%)

11. M - This is a code for a nonverbal move. The student actually makes a table, draws a diagram, writes, or calculates on paper. This usually occurred with no verbal statement (M) preceding it. (4, .2%)

In general then, these 11 "heuristics" were used VERY little across all students. However, it still seemed interesting to compare these results with those for the high scoring students. The percents for them are given in table 1. Also given in parentheses are the total number of processes and the number of times each specific one was used by that student.

In analyzing table 1, some interesting results occur. For these five students, which account for 16.6% of the 30 students, the following statistics can be gleaned from the table.

* An was used by them 2 out of the 4 times (or 50%).

Ag was used by them 6 out of the 27 times (or 22%).

P was used by them 5 out of the 25 times (or 20%).

* Da was used by them 4 out of the 10 times (or 40%).

G was used by them 9 out of the 33 times (or 27%).

* I was used by them 3 out of the 4 times (or 75%).

M was used by them 1 out of the 4 times (or 25%).

So it seems that (for this problem) these five students as a group use the heuristic-type processes of making analogies, noticing that data are needed, and making a generalization after inductive reasoning consider-

TABLE 1

Student (number of processes for Polyhedra)

	8 (71)	11(110)	20(109)	28(72)	29(84)	overall(1914)
An	1.4(1)	0	0	0	1.2(1)	.2(4)
Ag	0	0	2.8(3)	2.8(2)	1.2(1)	1.4(27)
P	0	.9(1)	2.8(3)	0	1.2(1)	1.3(25)
Da	0	1.8(2)	1.8(2)	0	0	.5(10)
R	0	0	0	0	0	.2(4)
D	0	0	0	0	0(1)	0(1)
G	0	1.8(2)	.9(1)	4.2(3)	3.6(3)	1.7(33)
I	1.4(1)	.9(1)	0	1.4(1)	0	.2(4)
A	0	0	0	0	0	0(?)
M	0	0	0	0	0	0(?)
M	0	.9(1)	0	0	0	.2(4)

ably more than the other 25 students. In comparing these five students to each other, it seems that student 20 used planning (P) more than the other four students. Also note that student 29 was the only one to use "D" (posing a theory and then trying to see if that theory is true).

HEURISTICS AND MATHEMATICAL CONTENT

The next heuristics used for analysis relate to the organization scoring system used. Each student was scored as shown in table 2. A student received the most points for the "a" categories and the least points for the "c" or "d" categories. The mathematical content was also analyzed (for the Polyhedra problem) to see if certain heuristics elicited different mathematical responses. The mathematical content was divided into categories A-H as described here.

A. The student gives some relationship between faces, vertices, or edges on ONE polyhedron. Examples: The student counts the number of

TABLE 2
ORGANIZATION SCORING SYSTEM

(Student receives points only when steps are in sequence.)

1. Poses a general problem
   a. poses all specific problems needed to solve and gives conclusion after solution
   b. poses some specific problem (at least 1)--doesn't necessarily solve
   c. poses none of the specific problems but solves or tries to solve the general one
2. Specific problem solved (score separately for each)
   a. needs > 2 steps to solve
   b. needs 2 steps to solve
   c. needs 1 step to solve OR student explains procedure for solving
3. Descriptive guesses (covers nonverbal moves in combination with verbal moves also)
   a. student covers all possibilities of what he is doing; tries all combinations
   b. student does $\geq \frac{1}{2}$ of all possibilities or combinations (where $2\frac{1}{2}$, round to 3)
   c. student does $< \frac{1}{2}$ (but > than 1 move) of all possibilities
4. Trial-and-error or pattern search
   a. gives all in pattern and follows through to generalization
   b. gives all in pattern OR gives ≥ 3 in pattern and generalization
   c. gives ≥ 3 in pattern
   d. gives < 3 but ≥ 1 in pattern but also gives generalization

faces. The student describes the type of polygon comprising the faces. The student describes the edges as congruent on 1.

B. The student gives some relationship between faces, vertices, or edges on more than one polyhedron. Examples: The student says that 1, 2, and 4 have congruent triangles. The student says that the edges of 3 are equal to the edges of some of the other polyhedra.

C. The student dissects one of the polyhedra or imagines composing one from the others. Examples: The student tries to imagine how you could use 1's to build a 2 or 2's to build a 4 or 1's to build a 4. The student imagines cutting 2 into square pyramids and putting one of them together with 3 to make something.

D. The student uses the polyhedra for tessellating space. Example: The student says that an infinite number of cubes could be put together without there being holes in the design.

E. The student uses the idea of symmetry. Examples: The student talks about splitting the faces on 1 in half. The student talks about cutting each polyhedron in half because each is symmetrical.

F. The student uses the relationship of volume or surface area or heights. Examples: The student says that 4 and 5 have the same volume. The student says that 4 and 5 are the same height.

G. This is a miscellaneous category for other geometric properties. Examples: The student says that each has a face parallel to the table when you set it down, except for number 1. The student talks about the "rollability" of number 4. The student talks about stacking the polyhedra on top of one another.

H. The student uses the polyhedra for other mathematical purposes. This category was used only once when student 29 talks about using 5 as a die and talks about probability.

The analysis of the number of times a specific content category was

used for a specific heuristic (from table 2) is given in table 3. Each denominator of a fraction indicates the total number of occurrences (across all 30 students) of that content area. Each numerator indicates the number of times a particular heuristic was used for that content category. For example, 6/24 for (2c, A) means that heuristic 2c was used 6 times (out of 26) with content category A (out of the 24 times that A was used across all heuristics).

In analyzing table 3, the most used heuristics across all content were 2c, 3a, and 3c. When each of these heuristics is analyzed for content it is seen that for 2c, out of the 26 times it was used, content categories A, B, and C account for 19 (or 73%) of those times. For 3a out of the 25 times it was used, content areas A and G account for 17 (or 68%) of those times. For 3c, content was more evenly divided among categories A, B, C, and G. So it seems that for solving a one-step problem (from 2c), the content elicited was giving relationships between faces, vertices, or edges on one or more of the polyhedron and dissecting one of the polyhedra. For (3a) trying all combinations of guesses, the content elicited most was giving relationships on one polyhedron and some miscellaneous properties.

Other interesting results from table 3 are that 3b was spread across five categories but 50% of its use was in content area B (relationships between more than one polyhedron). Similar results occurred for 4b.

When the table is analyzed for the content, the most used areas were A, B, C, and G. When looking at category A, it is seen that it was elicited most by 3a (37½%) as compared to 2c and 3c (25% each). When looking at category B, it is seen that it was elicited most by 4b (20½%), 3b (17½%), and 2c & 3c (14½% each). Category C was elicited most by

TABLE 3

	A	B	C	D	E	F	G	H	Heuristic Totals
1b			2/23				1/24		3
1c		3/34	1/23				2/24		6
2a			2/23			1/9			3
2b		3/34	1/23				1/24		5
2c	6/24	5/34	8/23		2/9		3/24	2/2	26
3a	9/24	1/34	2/23	1/2		4/9	8/24		25
3b	1/24	6/34			1/9	1/9	3/24		12
3c	6/24	5/34	4/23			1/9	4/24		20
4a	1/24	2/34			3/9	1/9			7
4b	1/24	7/34	2/23	1/2	2/9	1/9			14
4c			1/23		1/9		2/24		4
4d		2/34							2

Content codes:
A - relationship on one polyhedron
B - relationship between two or more
C - dissecting or composing
D - tessellating
E - symmetry
F - volume, surface area
G - misc. properties
H - uses for other purpose

Heuristic codes:
1's - posing a general problem
2's - solving a specific problem
3's - descriptive guessing; trying all combinations
4's - trial-and-error, pattern search, giving generalizations

2c (35%) and category G was elicited most by 3a (33 1/3%). From these most used content areas then, it seems that what stands out is dissecting or composing (C) being elicited by solving a specific problem in one step. Also notice that even though category F was used only 9 times, four of those times (44%) were elicited by 3a (giving all the possible results or combinations).

These heuristics and content categories were then analyzed for the "top 5" students. In table 4 each heuristic is given along with each student. In the body of the table, each time a content category was used, it is listed.

In looking at the heuristic totals, they seem to be fairly evenly divided among 2c, 3a, 3b, 3c, 4a, and 4b. When content totals are found (across all five students), symmetry was used the most, followed by relationships between more than one polyhedron, dissecting or composing, and relationships on one polyhedron. When each of these areas was analyzed, it was found that symmetry (E) was used 60% (3 of the 5 times) of the time in heuristic 4a. Also important is that relationships between more than one polyhedron (B) was used 60% (3 of the 5 times) of the time in heuristic 3b.

When the students are analyzed individually, interesting results occur. Heuristic 4c was only used 4 times across all students, and student 8 used it 2 of those times. Heuristic 1b was only used 3 times across all students (see table 3) and student 20 used it 2 of those 3 times. Similarly heuristic 4d was used a total of twice and student 28 used it one of those 2 times. When looking at the content categories, it is seen that A was used 2 of the 5 times by student 8 and 3 of the 5 times by student 20. C was used 4 of the 6 times by student 20. Thus it seems that some heuristics and content are particularly attributable to these five students. This is seen when comparing tables 3 and 4.

TABLE 4

	8	11	20	28	29	Heuristic totals
1b			C,C			2
1c						0
2a						0
2b						0
2c	A		A,E		H,H	5
3a	D		A	C		3
3b		B,B	E	B,G		5
3c	A		A,C,C		B,F	6
4a	E	F		E	B,E	5
4b		D,E	B	E		4
4c	C,E					2
4d				B		1

Content totals:

A - 5	E - 8
B - 7	F - 2
C - 6	G - 1
D - 2	H - 2

These five students, which account for 16.6% of the total number of students, used heuristic 1b two of the three times (or 66 2/3%); used heuristic 3b five of the 12 times (or 41.7%); used heuristic 4a five of the seven times (or 71.4%); used heuristic 4b four of the 14 times (or 28.6%); used heuristic 4c two of the four times (or 50%); and used heuristic 4d one of the two times (or 50%).

As mentioned previously, these students seemed to be content specific in the use of symmetry (E). Category E was used by them eight of the nine times (or 88.9%). Also interesting was the disuse of category G (miscellaneous properties) by these students. They used it only once of the 24 times (or 4.2%).

OTHER ANALYSIS FOR THE "TOP 5" STUDENTS

Other Heuristics

The transcript for the Polyhedra problem for each of students 8, 11, 20, 28, and 29 was further analyzed. See the Appendix for a copy of each transcript.

In each, examples that related to Polya's problem solving plan were sought. In particular, examples were sought where students worked backwards, used analogous problems, checked their results, or varied a problem. Also looked for was any indication of a general plan of solution as Polya specified, i.e. devising a plan, carrying it out, and looking back. Examples were also sought that would relate to the Gestalt-like heuristics of evidence of tension, its resolution, and "insight" in solving a problem. (This especially relates to the work of Wertheimer, 1959.) A discussion of this analysis is given by student.

Student 8. When the student speaks of how you could use each of the polyhedra to make a design without there being any holes (tessellating

space), this is a variation on example 2 given in the directions to the students. (See Appendix for a copy of the directions.) Next he discusses how some of the polyhedra can be split in half. He evidences some tension in that he is not sure about 4 and 5. After a pause, he appears to have some "insight" and says that you could split 4 in half. Later, he varies this idea and talks of splitting the faces on a polyhedron in half. Both of these "splitting" ideas are variations of example 4 in the student directions.

Student 11. When he discusses how a pyramid could be formed if you put 1's on each face of 2, there seems to be a moment of "insight" when he pauses and then says that you'd need 8 of them (referring to 1's). Next, he varies the first idea by saying that 4 has 20 times the volume of 1. He also checks his work by recounting the number of faces on 4. Next, he speaks of tessellating space which is a variation of example 2 in the directions. Later he speaks of cutting 2 in half and then, by induction, says that you can divide all of the polyhedra in half because they're all symmetrical. This is a variation of example 4 in the directions, where the idea is extended to three dimensions. Next, he varies a previous idea by stating that 2 is 8 times the volume of 1.

Student 20. The first section is a long one and basically deals with the problem of trying to count the number of faces on 4. When he wonders if the counting idea could be used to help count the number of faces on 4, it relates to using an analogous problem to solve another problem. The student then gets frustrated; he counts the faces on 2 and 3; and he goes back to 4 again. He is trying to use analogous ideas. There is evidence of tension when the student says he is "so discouraged" but the tension is not resolved and the student gives up on the particular

problem. In the next section, he tries to put 1 and 3 together in a variation of the idea of tessellating space (also a variation of example 2 in the directions). Next, he discusses the kinds of faces comprising each polyhedron. Then he notices that all the triangles on 1, 2, and 4 are congruent. Next, a problem is posed, which is a variation of putting 1 and 3 together. Some tension is evident because the student says he's "not sure how 2 is structured"; the tension is not resolved; the student progresses and varies the problem by speaking of fitting 4's together to make a larger object. Then he returns to the first idea of counting faces. He recounts the number of faces on 2 but now varies the problem by trying to determine the name of 2. Tension and conflict are evident because the student says that there are eight planes and he knows that eight refers to an octagon. However, the object doesn't look like an octagon. This time the conflict is resolved because the student realizes that the difference lies in the fact that the object has eight planes not sides. He then varies <u>this</u> problem by starting to count the number of edges on 2 and then on 1.

<u>Student 28</u>. This student uses one idea, that of "same angles on the outside", and varies its usage in the first few sections of the transcript. He first talks of 4 being made of triangles like 1 because of the angles on the outside; then he talks of 2 being composed of 1's because of the angles; and then he talks of 1, 2, and 4 being related because of the "angles of the lines on the outside". Later he talks of each polyhedron being able to "set" on the table and having a face parallel to the table. He then varies this idea by discussing parallel faces on a polyhedron. Another variation is next when he speaks of a polyhedron being able to "set" on a point so that it could spin. In the

last section, he talks of splitting 5 in half and says that you can do that for all the polyhedra. This involves inductive reasoning and is a variation on example 4 in the directions.

Student 29. In the first section, this student discusses how you could use 5 as a die and if there was a number on each face you'd have one chance in 12 of throwing that number. He then varies that idea and tells what would happen if two of the faces had the same number. Next he varies example 2 of the directions and speaks of fitting 1, 2, and 4 together. Next, he counts the number of faces of 2; he varies this idea by telling what would happen if you cut 2 along the "crack" so then you'd have a total of 10 faces; and then he counts the faces of 3 and compares the results to 2. Next he uses the comparison idea to say that 4 is taller than 5 and thus has more volume. Using analogous reasoning, he says that 3 is taller than 2 and thus it has more volume.

Piaget Analysis. To determine if any of the characteristics of formal operations were present (such as finding all combinations or inductive or deductive thought, re: Inhelder & Piaget, 1958), tables 1 and 4 can be used. From table 1, it is seen that students 8, 11, and 28 used induction for this problem whereas students 20 and 29 did not. They also used it more than the other students. (It should be explained here that many times students used deductive or inductive reasoning but were only coded D or I if they reached a final generalization by that reasoning process.) To look at the use of finding all combinations, table 4 can be consulted. Essentially heuristics 3a, 4a, and 4b cover this idea. Students 8 and 29 used these heuristics 1/3 of the time; students 11 and 28 used them 3/5 of the time; and student 20 used them 1/5 of the time.

Comparison of Students

Mathematical topics in common. Each of these five students used the concept of symmetry but sometimes in different ways. Student 8 explicitly speaks of the plane or line passing through any vertex, implying that he is considering more than one plane or line of symmetry. He was the only student to consider line symmetry for faces of the polygons. Student 11 first implicitly uses plane symmetry when speaking of cutting 2 and then explicitly generalizes to all the polyhedra. It is not clear whether or not he is considering a polyhedron as having more than one plane of symmetry. Student 20 only implicitly uses plane symmetry to help her count the number of faces of 4. Student 28 uses both plane and rotational symmetry implicitly. Student 29 uses plane symmetry both implicitly and explicitly. His focus is similar to student 11 in that he concentrates on the plane passing only through the edges of a polyhedron. However he explicitly uses this idea whereas student 11 uses it implicitly.

The topic of composition is another one these students have in common. Student 8 thinks of 4 as being composed of 2's or 1's and of 2 as being composed of 1's. Student 11 thinks of composition in a different way. He is trying to compose a new shape by attaching 1's to 2. Student 20 uses composition in a similar way to student 8 by considering 2 and 4 as each composed of 1's. She however does not consider the possibility of 4 being composed of 2's. Student 20 also uses composition similar to student 11 by thinking of a new shape built from 4's. Neither student 28 nor 29 use the composition idea.

Tessellation of space is used by students 8, 11, 20, and 29. Student 8 considers each polyhedra, except number 1, separately. Student 11 does the same except he considers number 1 also. He and student 8 conflict as to whether or not polyhedron 4 could tessellate. Student 20 uses the tessellation in a different way by considering the combination of

1, 2, and 3. Student 29 uses tessellation in a way similar to student 20 by combining 1, 2, and 4 to say that they could fill space together but that 3 and 5 could not.

Students 11, 20, and 28 use the idea of congruence in relation to the faces on polyhedra 1, 2, and 4. The idea of counting the number of faces on a polyhedron is used by students 11, 20, and 29. However each student focusses on a different aspect. That is, student 11 uses counting to make the comparison that 4 has twice as many faces as 2. He then uses counting in a more "elegant" way by relating the number of faces on 1 to the number of sides of a face on 3. Student 20 simply focusses on the counting of the number of faces. Student 29 uses counting in connection with the symmetry idea as a way of comparing object 2 with new objects formed.

Area and/or volume is used by students 11 and 29. Student 11 uses area and volume in many instances. He compares the area and volume of 4 and 1. He uses volume along with the symmetry idea to compare the volume of the two pyramids formed from object 2. He compares the volume of 2 and 1. In the last portion of his transcript he compares the volumes of all the polyhedra. Student 29 compares volume for polyhedra 4 and 5 and then for 2 and 3. However he does not extend his comparison to all the polyhedra as student 11 did.

Reoccurrence of topics or ideas. A topic or idea is considered as reoccurring if it occurs in some later section of the student's work after if first appears. For students 8, 28, and 29, symmetry is the reoccurring idea. Student 8 first uses symmetry three dimensionally and then later two dimensionally. Student 28 uses two different types of symmetry, rotational and plane. Student 29 uses plane symmetry implicitly and later explicitly. Student 11 has several reoccurring ideas: (a) the relation

between the faces on 1, 2, and 4; (b) volume; and (c) "cutting in half".

Dominance of 3-dimensional vs. 2-dimensional thought. Student 8 shows a dominance of 3-dimensional thought, with a shift only once to 2-dimensions. Student 11 shifts back and forth between the two and does not show any dominance of one over the other. Student 20 only uses 2-dimensional thought occasionally; uses 1-dimension when counting edges of a polyhedron; and shows a dominance of 3-dimensional thought. Student 28 always uses 3-dimensional thought except for the one occasion he uses 1-dimension when comparing lengths of edges of polyhedra. Student 29 uses 1-dimension when comparing heights, but otherwise shows a dominance of 3-dimensional thought.

Mathematical elegance. In his study of gifted children in the Soviet Union, V. A. Krutetskii (1976) found many characteristics. As described by Kilpatrick and Wirszup in the introduction to his book, two of these characteristics were: striving for an elegant solution and having a mathematical cast of mind. In a sense, I am combining these two characteristics into one and calling it "mathematical elegance." This characteristic is hard to define because of its subjective nature but evidence of it occurred in these students. Some examples are: (a) when student 11 relates the faces around a vertex on 4 to a pentagon and polyhedron 5; (b) when student 11 relates the number of faces on 1 to the number of sides of a face on 3; (c) when student 11 relates polyhedra 2 and 3 through projective geometry ideas; and (d) when student 20 tries to count the faces on 4 by using the fact that since 5 can be cut in half along a zigzag line maybe 4 can be also.

Summary

Obviously more comparisons could be made for these students. I have only included some of the most interesting ones here. Although these comparisons and other analysis showed that these five students had much in common (for the Polyhedra problem), it seems that overall each student was highly individualistic. Probably a similar detailed analysis of the other 25 students would show some sharply contrasting styles and use of content and heuristics.

Some reasons for not finding more prevalence of Polya-like, Gestalt-like, or Piaget-like heuristics could be: (a) since the students were asked only to make conjectures, they would most likely not follow an entire problem solving procedure; (b) if the method of interview would have been more interactive, probing, and questioning then perhaps more information would have been obtained; (c) if the original purpose of the study had been to look for the Polya, Gestalt, and Piaget heuristics, then the study could have been conducted differently and data could have been handled differently so more information would have been obtained; and (d) the problem solving process is difficult to study in one hour's time since frequently "insight" occurs many days or weeks after an initial idea.

IMPLICATIONS FOR EDUCATION

Implications that can be drawn from the work of the Gestaltists are: (a) student's spatial perceptions may be vastly different from adult perceptions and in this respect teacher awareness is important; (b) teachers could discuss "insight" and any of their experiences relating to getting a sudden solution to a perplexing problem; and (c) time for problem solving and "insightful" thinking needs to be

allowed for.

Central to Wertheimer's and Piaget's work is the idea of the probing interview. By using such interviews, the teacher can learn most what the student understands or what he does not, and WHY the student understands some topic or WHY he does not. A recently translated article, The Clinical Method in Education (Denis-Prinzhorn & Grize, 1966) has many useful suggestions for teachers in this regard.

The ideas such as reoccurrence of ideas, dominance of 3-dimensional vs. 2-dimensional thought, and mathematical elegance are important ones for teachers to consider. Such behaviors, when noticed in students, could be the basis for some discussion among teacher and students.

In reference to some of the results from this further analysis of the data, I am not sure what implications could be made for education. No general conclusions can be made on such a small study. Even drawing any specific conclusions would be presumptuous. The purpose of looking at how excellent mathematics students think is not to further categorize and separate students. Rather, we need to get some insight into what these students do and how they behave that is different from other students. For example, here it was shown that (in reference to their heuristics) the "top 5" students used mathematical ideas, such as symmetry, completely different from other students. Perhaps this only shows how individualistic all students are; perhaps this shows something unique about these five students; or perhaps this shows nothing at all. More importantly, it shows the need for more research in this area of creative thinking.

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DIRECTIONS

Each problem you will work on DOES NOT ask for a specific answer. In fact each problem was designed so that you can give many answers.

The problems involve your making mathematical guesses. To make your guesses you can look for relationships or patterns; look for likenesses and/or differences; base your guess on changing or adding something to the problem; or guess something you think is true. These are just examples of what you could do. You can make other types of guesses but remember to make them mathematical.

Let's do an example to make this clearer. Suppose you have a regular pentagon, . Several guesses are:

1. Since the sides are equal, the angles are equal.

2. If you had many such pentagons, you could not place them side by side without there being "holes" in the design.

3. If a dot is placed in the center and triangles are drawn like each triangle would be congruent, or equal.

4. If a line is drawn from a vertex and perpendicular to the opposite side like , it divides the pentagon into two equal parts and the area of each part is half the area of the pentagon.

5. If lines are drawn from a vertex to separate the pentagon into triangles, you get three of them like whereas in a hexagon you get four of them .

Can you make a guess?

You will have 5 problems to work on. If you run out of guesses for a problem, go on to the next. I will remind you if you are spending too much time on a problem. Any questions?

Example of Recording Technique

To help the reader with interpreting the transcripts, an example is given below of how I recorded the student's physical moves with other notes so they could be interwoven when the tape was transcribed. All physical moves are recorded on the left with key words in quotes directly to the right of the move. Sometimes the student was silent while making a physical move so a blank space is to the right of that particular move. The column on the far right provides the reader with an explanation of the abbreviations I used in the other two columns.

PO 1		working on polyhedra
Ma 4		manipulates icosahedron
Ma 5		manipulates dodecahedron
cts faces 5	"on 12 sided ... 1 out of 12 chance"	
1 on 2		puts tetrahedron on octahedron so faces match
1 on 4		
2 & 1 on 4	"1, 2, 4 sides same shape .. fit tog."	both on icosahedron so that each is on a separate face
Ma 5 and 3	"3 & 5 can't tog. .. couldn't fit"	3 is the cube
Ma 2	"2 - 8 sided ... cut ... 5 sided pyramid ... be 10 .."	
Ma 4		
Ma 3		
Ma 4		
Ma 4 & 5		
cts 5		means counts faces
cts faces 4		
2 on 5		
Ma 1		

Ma 3		
Ma 2 & 3		
2 on 3	"2 similar to 3... both 8 sides"	
Ma 1 & 2	"1 .. split #2 .. no"	# is number
1 on 2		
4 next to 5	"area #4, 18 sided .. taller 4 > vol. 5"	places one next to the other on the table; volume
2 next to 3	"1kg at 2 & 3 ... 3 taller, larger vol."	looking
lines up 1 .. 5		places 1, 2, 3, 4, 5 in that order next to each other on table
1 on 2		
1 on 2 on 3		stacks 1 on top of 2 on top of 3

Transcripts

Polyhedra - Student 8

(Ma 3 & 4) Solid - there's a square and triangle. If put in patterns, (1 on 3) triangles would work and squares would work. (2 on 4) These are triangular sides for 2 and 4 -- 2 and 4 could probably put together in a pattern and probably wouldn't have any holes in it. (Ma 4) (Ma 5) 5 be a pentagon - wouldn't be able to do it, cause pentagon doesn't go together with 5 sided anyway so that wouldn't. (Ma 1) Split triangle in half by any one of its vertices - equilateral triangle - have same amount of area on half as other . (Ma 3) same with square. (Ma 2) same with number 2 -- square's number 3 and triangle's number 1 ... (Ma 4) not sure, think could do for just about all of them ... 1, 2, 3, 4, 5 - I'm not sure about 5 or 4. (Ma 4 and 5) 4 and 5, I'll think about for a while - I'm not sure because (pause) 4 would definitely cause if split it - triangle - would be triangles; triangles I know (1 on 4) could fit

a set number of triangles in there. If you take .. cut a triangle in half and have same area. (Ma 5) same for 5 too, could do same thing for pentagon. (2 on4) could fill number 4 with bunch of 2's. (Ma 4 &1) fill 4 with 1's too. (1 on2) 2 fill with number 1. (Ma 1) faces of them - number 1 all angles are same and if split in half, from one vertice perpendicular to opposite line, give two right angles - two right triangles - have two right angles of course and one be obtuse and other acute for other two angles. (Ma 3) Square give all right angles cause it's split into a rectangle - rectangle has all right angles anyway. (Ma 2) two all faces are same as 1. Can do same thing as that - split it and have two right triangles - faces inside - could ... take out triangles - could if split 2 in half along middle -- no along where it was cemented together, would give you - not sure what it's called, like the top of a house - a modern house - give you base with 4 sides and could find area of that - base times height by ½ or is that the triangle? I'm not sure. TRIANGLE don't care too much for geometry - this deals with geometry.. (Ma 3)

Polyhedra - Student 11

(Ma 2 &4; 2 on 4) Each face of 2 is the same as each face of 4 so they go together. (Counts faces on 4) OK there's I think 18 faces on 4 so there's no there's 10, 20 faces on this - so there's (2 on 4) more than twice as many (pause) (Ma 3 &4) (Ma 5) Each of these objects is different. (1 on 2) I venture to say if you had another triangle shape with 1 with 2, if put whole bunch of 1's on each face, you can get a triangle or pyramid using- you'd need 6 (pause) 8 of these so need 7 more to make a pyramid. (4 on 2 with 1) number 4 is 20 times the volume of 1 if it has 20 faces which it probably won't. (recounts faces on 4) should be 20 times the volume. (Ma 5) If put this number 5 with pentagons could put together and shapes would go - no they wouldn't. You could not put whole bunch of 5's together to make anything. (Ma 3) Could put 3's together to make a shape. (Ma 1 & 2) and

could put 1's and 2's - put 2's together. Could put (Ma 4) couldn't put 4 together. (Ma 2 on 3) (Ma 2, 4, 1) numbers 2, 4 and 1 alike cause faces are triangle shaped and (Ma 4 &5) 4 and 5 are alike cause shape of triangles put together on 4 is a pentagon which is the same as each face on number 5 . (Ma 3 & 2) (Ma 2) (Ma 1) (Ma 1 & 3) number 1 has as many faces as a square has sides. (Ma 5) (Ma 4) (Ma 2) If made a pyramid of number 2, cut in half and made two pyramids each would be half area, volume of 2 and would be and (Ma 1 & 2) each would be different from 1 cause they'd have 4 faces and 1 only has three faces. (Ma 5) same for everything, if cut in half - can't divide everything (Ma 2) only divide number 2 perfectly evenly with straight flat face .. you divide with. You can divide all of them if cut them directly in half cause they're all symmetrical. (Ma 3) (1 on 3) (1 on 4) (1 on 2) number 2 has (pause) 8 times volume of number 1 so it follows that (Ma 4 & 2) it would have (starts to write) no doesn't work. (Ma 2 & 3) If cut number 2 in half and made the two pyramids again each face of pyramid on bottom face of pyramid would be a square and would be same dimension as each of the faces of 3 so they fit together. (1 on 4) (Ma 3 & 2) put the - word.. side view - profile of number 2 is a square and is the same as 3 so that if you couldn't see depth or anything, it'd be the same. (Ma 1) (1 on 2) (Ma 3) (1 on 2) (1 on 4) I can ... (Ma 1 & 2) 2's bigger than 1 (Ma 3 &2) 3's bigger than 2. (Ma 5 & 3) 5's bigger than 3 and (Ma 4 & 5) I think 4's bigger than 5, I'm not sure it's hard to tell but it follows that 4's bigger than all, than any ... 4's bigger than 1, 4's bigger than 3, than 2 and 5's bigger than 1, 2 and 3 but not as big as 4 and 1's smaller than all of them.

Polyhedra - Student 20

(Ma 4) How many sides are these? (counts faces on 4) here's 5 here and here's another 5 ... so that's 15 plus 2 - guess 17 sides whereas number 5 (counts faces on 5) has -- if you cut -- no wait - yeh this is a weird one. This

has 1 -- 6 -- and could cut in half along zigzag line and get two equal parts. Let me see if this could happen with this. (Ma 4) I don't -- this couldn't be cut in half (pause) like that? 17's an odd number so couldn't be cut in half if it is 17 (counts faces of 4 again) can't draw on these? This is complicated to figure out by halves. (Ma 1) This is triangular shape - I mean a solid triangular - guess you call it a pyramid. (Ma 2) This is like a 6 sided - 12 sided - you could cut it in half and get two square pyramids with a square base coming up to a triangular object. This has 8 sides and this (Ma 3) has 4 sides. So 4, 8 wait - let me check that one again. (Counts faces on 4 again) so that's 10 in that area there - has 8 ... maybe it's 18 ... 10 in there then I get (counts) so discouraged on this one seems to be a line right here - maybe where it's glued together or something - a zigzag line. I see where you could cut it in half if you took -- that's why it's so hard to put your hand around it to count cause the lines go up and down so badly.. so (counts 4) that's 8.. this is 4 .. maybe 16 sides. (1 on 3) This is not one with hypotenuse or whatever - unequal hypotenuse - this doesn't really fit on the square in any even way. (Ma 1) If you put it's base - all sides could be bases and then up against (1 on 3) even thing so it's the .. it fits at an angle upwards - you could slip something in there. But if you took something like this, pretend that side isn't there (2 on 3) it would fit on fine and 2 and 3 have the same base - square. (Ma 5 & 4) Difference between 5 and 4 is that 4 contains triangles and 5 pentagons. This is many triangles. (Ma 4 & 2) 4 & 2 are many of pentagon - uh triangles put it together. 1 is also triangles and (Ma 3) 3 is made up of sides of squares. (1 on 4) These triangles -- 1 and 4 are same size triangles. (2 on 4) The sides of 2 and 4 are the same size triangles also and 1. All these triangles are the same size. So (1 on 2) if you fit all these triangles together could possibly -- what happen if put?

not sure what shape you'd come out with. I wonder if you had more of these triangles, could you make this object? If you had more 1's could you make number 2? (1 on 4) Pretty sure could make number 4. (Ma 2) Not sure how this is structured. (Ma 5) (5 on 3) (Ma 3) With these cubes you could - like I said you're pretty stuck with cubes making certain lines. (Ma 4) These things fitted on each other -- I think you could evenly fit these onto one another and maybe make a larger object. (Ma 2) I'm not sure what object 2 is shaped like - I think it's a diamond - no particular name for it. (Ma 4 & 5) (Ma 4 & 2) (Ma 2) an octagon - is an octagon? This has 8 sides but doesn't look like typical octagon. An octagon looks like one of these things, (Ma 5) with 8 sides. (Ma 2) These have 8 parts - I mean 8 planes to them I guess -- that's the difference - 8 planes not 8 sides -- lots of sides -- (counts edges of 1) The triangle has 6 edges. (counts edges of 2) This has 12, even though 2 is the same plane as 1 it has the same planes all over it. (Ma 1 & 2)

Polyhedra - Student 28

(Ma 2) (Ma 1) (1 on 3) (1 on 2) (1 on 3) (1 on 5) (1 on 4) number 4 seems to be made up of small triangles like 1 -- all fitted together - that's the type of shapes on outside - same angles for lines on outside. (Ma 3) These are all have depth - three dimensional. (Ma 2) number 2 seems to be two pyramids glued bottom to bottom to make this octagon. (1 on 2) looks like it's made up of little 1's, many 1's also cause has the same angles on the outside. (1 on 2) (Ma 4) looks like numbers 4, 1, and 2 are related cause of angles of lines that make up the outside. (Ma 5) 5 doesn't seem to be related to any of these because the length of the sides of - that are equal but not equal to any of the other ones. (1 on 3) all other ones have equal sides except number 5. (Ma 3) you could cut 3 into - if cut it, cut it either into -- flat triangles no depth or could

cut into -- or it's made up of rectangles, so rectangles, several other smaller squares (Ma 4 & 5) all have flat sides and lines aren't curved (stacks 5 on 4 on 3) all seem to have a top and bottom so if set it flat on table have a flat top parallel to that so there's a - no matter which way set it (Ma 5) have two sides of each of things that are parallel in length away from each other and are same size, (Ma 1) except for the triangle which doesn't have a top and bottom. (1 on 2) (2 on 4) (Ma 2 & 5) (Ma 2 & 3) (Ma 2 & 4) (Ma all) All of them you could set on a specific point if you set it so that another point is so - it's straight up so could spin around except for the triangle. The triangle doesn't seem to fit with other shapes cause doesn't have sides that are parallel to it. (Ma 3) (looks at all) (lines them up from left to right: 4, 2, 3, 5, 1) (Ma 5) If split 5 in half it would - no matter which way split since it's - cause it looks as if it's almost equal - split in half get two equal sides, same shape. (Ma 3) same with the square (Ma 2) and all the rest except for the triangle again.

Polyhedra -- Student 29

(Ma 4) (Ma 5) (counts faces 5) on 12 sided figure, if wrote a no. on each face & rolled it have a 1 out of 12 chance to get a certain no. if they were all different. If had 2 the same it'd be 2 out of 6 or 2 out of 12 or 1 out of 6 chance & so on. (1 on 2) (1 on 4) (2 & 1 on 4) Figures 1, 2 & 4 sides are same shape & exact same size for all three so can fit those together. (Ma 5 & 3) But 3 & 5 can't put together - wouldn't fit together right - because 3 has 4 sides & 5 has 5 sides. (Ma 2) #2 it's 8 sided. If were to cut right down the middle along crack there, get a 5 sided pyramid shape, 5 being the bottom. The total number of sides - of new shape would be 10 which would be 2 more than sides of this cause you'd be making 2 new sides by splitting it. (Ma 4)

(Ma 3) (Ma 4) (Ma 4 & 5) (counts faces 5) (counts faces 4) (2 on 5) (Ma 1) (Ma 3) (Ma 2 & 3) (2 on 3) #2 similar to #3 because both have 8 sides & #2 looks like a square that's elongated - a diamond shape. (Ma 1 & 2) #1 if split #2 as said before - no scratch that. (1 on 2) (4 next to 5 on table) Since area of #4, the 18 sided, since taller than #5 & it's got more sides which makes it closer to a circle, then 4 has more volume than 5. (2 next to 3) looking at #2 & 3. Since 3 is taller & they have same no. of sides, 3 has a larger volume. (lines up 1 thru 5) (1 on 2) (1 on 2 on 3) (1 on 2 on 4) (Ma 2 & 4) #2 only figure that could split without bothering any of the planes - split in half because can go all way around on same line - could split on that line without bothering any of the planes. Doesn't work for any of other shapes. (Ma 5) (Ma 5 & 2) If did that to any other lines, have to have a jagged line. If split the 12 sided figure like that right in half by going along edges of planes have to make 10 - go in 10 different directions - 10 different cuts. (Ma 4) (Ma 3 & 5) #3 & #5 are -- #5 isn't - #3 only shape here that has 2 ≅ sides on it. (1 on 3)